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On the representation of distributive algebraic lattices. III

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1. Introduction

Around 1980, unpublished investigations of Heiko Bauer led to the conjecture that every distributive semilattice with 0 of cardinality $\leqq \aleph_1$ is isomorphic to the semilattice of compact congruences of a lattice. In [2], H. DOBBERTIN gave a partial ordering which can be used to prove that on a set of cardinality $\aleph_1$ there is a directed family of finite subsets covering every finite subset such that the Boolean lattice 2^3 is not order-isomorphic to any subset of this family. We shall use this fact to prove the above formulated conjecture.[2]) In more usual terms, this means that every algebraic lattice with 0 having at most $\aleph_1$ compact elements is the congruence lattice of a lattice. We cannot extend the proof for more than $\aleph_1$ compact elements, the reason for that will be discussed in [—][3]). Note that the case of finitely many compact elements was already settled in [3], while the countable case was discussed in [2] and [5].

[1]) This paper was left behind by András Huhn in the form of a first draft of a manuscript. The remarks in footnotes 2, 3, 4 and 6 are due to Hans Dobbertin, who was kind to prepare the paper for publication.

[2]) In [1] the mentioned partial ordering has already been used implicitely in order to prove a theorem (see [1; Thm. 3.4]) which has the following corollary: *every distributive semilattice with 0 of cardinality $\leqq \aleph_1$ is the image of a generalized Boolean lattice under a weak-distributive $\vee$-homomorphism.* (See [6] for the definition of the notion "weak-distributive".) In the present paper a sharper result is proven, namely "weak-distributive" is replaced by "distributive". The important new idea in András Huhn's proof is the use of "reduced free products".

[3]) Perhaps András Huhn had planned to write another paper to which he made a reference here, but unfortunately no manuscript of it has been found in his inheritance. It is also possible that he wanted to make a here reference to [2].

2. Outline of the proof

Let D be a distributive semilattice with 0. Assume that $|D|=\aleph_1$. First we define a directed family of finite subsets of D. Let $\alpha<\omega_1$ be an ordinal number. For $\alpha=0$, let $h_\alpha=\{0\}$, where 0 is the lower bound of D. For $\alpha=n+1$ ($n\in\mathbf{N}$, $\mathbf{N}$ denotes the set of natural numbers), let $h_\alpha=h_n\cup\{a\}$, where $a\in D\setminus h_n$. Now if $\alpha=\omega\beta+n$, $n\in\mathbf{N}$, then we proceed as follows. $\omega\beta$ has a confinal ω-chain $\alpha_0<\alpha_1<\dots$. For $\alpha=\omega\beta$, let $h_\alpha=h_{\alpha_0}\cup\{a\}$ with $a\notin h_\gamma$ for $\gamma<\omega\beta$. For $\alpha=\omega\beta+n+1$, let $h_\alpha=$ $=h_{\alpha_{n+1}}\cup h_{\omega\beta+n}\cup\{a\}$ with $a\notin h_{\omega\beta+n}$. Let H be the set of all h_α, $\alpha<\omega_1$. The inclusion relation orders H, this ordering will be denoted by $\leqq$. h_0 will also be denoted by 0.[4])

Figure 1 shows how $\{h_\gamma\colon \gamma<\omega(\beta+1)\}$ is constructed from $\{h_\gamma\colon \gamma<\omega\beta\}$.

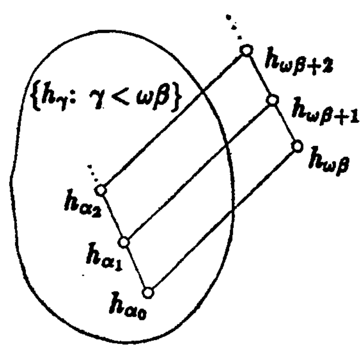

Figure 1

For every $h\in H$, choose a finite distributive 0-subsemilattice D_h of D such that $h\leqq k$, $h, k\in H$, implies $D_h\subseteq D_k$. This can be carried out by induction on α, using the fact that any finite subset of D is included in a finite distributive 0-subsemilattice. Then D is the direct limit of the D_h's. For later purposes we introduce the notation

$$\varepsilon(h,k)\colon \begin{cases} D_h \to D_k \\ d\mapsto d \end{cases}$$

provided that $h\leqq k$ (and therefore $D_h\subseteq D_k$).

[4]) The definition of the family $(h_\alpha)_{\alpha<\aleph_1}$ has to be modified slightly in order to guarantee that D is in fact completely exhausted.

Now, for every $h, i \in H$ with $h \leqq i$, we shall define finite distributive lattices $D(h, i)$ and 0-preserving lattice embeddings

$$\varphi(hg, i)\colon D(h, i) \to D(g, i) \quad \text{for} \quad g \leqq h \leqq i,$$
$$\varphi(h, ij)\colon D(h, i) \to D(h, j) \quad \text{for} \quad h \leqq i \leqq j$$

such that the following diagrams be commutative

(1)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] \arrow[d, "\varphi(hg, i)"'] & D(h, j) \arrow[d, "\varphi(hg, j)"] \\ D(g, i) \arrow[r, "\varphi(g, ij)"'] & D(g, j) \end{tikzcd}$$

(2)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(hg, i)"] \arrow[dr] & D(g, i) \arrow[d, "\varphi(gf, i)"] \\ & D(f, i) \end{tikzcd}$$

(3)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] \arrow[dr] & D(h, j) \arrow[d, "\varphi(h, jk)"] \\ & D(h, k) \end{tikzcd}$$

where $g \leqq h \leqq i \leqq j$, $h \leqq i \leqq j \leqq k$, respectively. We denote by $B(0, i)$ the smallest Boolean extension of $D(0, i)$ and by $\chi(0, i)\colon D(0, i) \to B(0, i)$ the canonical embedding (precisely defined later). We also define 0-preserving lattice embeddings $\psi(0, ij)\colon B(0, i) \to B(0, j)$ for $i \leqq j$ such that the following diagrams are commutative

(4)
$$\begin{tikzcd} B(0, i) \arrow[r, "\psi(0, ij)"] \arrow[dr] & B(0, j) \arrow[d, "\psi(0, ij)"] \\ & B(0, k) \end{tikzcd}$$

(5)
$$\begin{tikzcd} D(0, i) \arrow[r, "\chi(0, i)"] \arrow[d, "\varphi(0, ij)"'] & B(0, i) \arrow[d, "\psi(0, ij)"] \\ D(0, j) \arrow[r, "\chi(0, j)"'] & B(0, j) \end{tikzcd}$$

for $i \leqq j \leqq k$ and $i \leqq j$, respectively. Now let $B(0, -)$ be the direct limit of all $B(0, i)$ and $D(h, -)$ be the direct limit of all $D(h, i)$. Using the above commutativities, we can define embeddings

$$\varphi(hg, -)\colon D(h, -) \to D(g, -),\quad g \leqq h,\quad\text{and}\quad \chi(0, -)\colon D(0, -) \to B(0, -)$$

such that the following diagram is commutative

$$\begin{tikzcd} D(h, -) \arrow[r, "\varphi(hg, -)"] \arrow[dr] & D(g, -) \arrow[d, "\varphi(gf, -)"] \\ & D(f, -) \end{tikzcd} \tag{6}$$

The mappings $\varphi(hg, i)$ have inverses $\varphi'(gh, i)$ (which means that

$$\varphi(hg, i)\varphi'(gh, i) = \mathrm{id}_{D(h, i)},$$

the mappings are carried out in the written order) such that the following diagrams are commutative with $g \leqq h \leqq i \leqq j$ and $f \leqq g \leqq h \leqq i$, respectively.

$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h, ij)"] & D(h, j) \\ D(g, i) \arrow[u, "\varphi'(gh, i)"] \arrow[r, "\varphi(g, ij)"'] & D(g, j) \arrow[u, "\varphi(gh, j)"'] \end{tikzcd} \tag{7}$$

$$\begin{tikzcd} D(h, i) & D(g, i) \arrow[l, "\varphi'(gh, i)"'] \\ & D(f, i) \arrow[ul] \arrow[u, "\varphi'(fg, i)"'] \end{tikzcd} \tag{8}$$

The right inverses are monomial 0-preserving weakly distributive $\vee$-homomorphisms (in the sense of SCHMIDT [6]). Also the $\chi(0, i)$'s have 0- and $\vee$-preserving monomial weakly distributive right inverses $\chi'(0, i)$ and we have the following commutativities

$$\begin{tikzcd} D(0, i) \arrow[d, "\varphi(0, ij)"'] & B(0, i) \arrow[l, "\chi'(0, i)"'] \arrow[d, "\psi(0, ij)"] \\ D(0, j) & B(0, j) \arrow[l, "\chi'(0, j)"] \end{tikzcd} \tag{9}$$

for $i \leqq j$. These commutativities allow to carry over the $\chi'(0, i)$ to the direct limit $B(0, -)$ and so we get a 0-preserving monomial weakly distributive $\vee$-homomorphism

$$\chi'(0, -)\colon B(0, -) \to D(0, -)$$

which is a right inverse to $\chi(0, -)$. Similarly, we define $\varphi'(gh, -)$ for $g \leqq h$.

$$\varphi'(gh, -)\colon D(g, -) \to D(h, -)$$

is again a 0-preserving monomial weakly distributive $\vee$-homomorphism and a right inverse to $\varphi(hg, -)$. Again we have that the following diagram is commutative

$$\begin{tikzcd} D(h, -) & D(g, -) \arrow[l, "\varphi'(hg, -)"'] \\ & D(f, -) \arrow[ul] \arrow[u, "\varphi'(gf, -)"'] \end{tikzcd} \tag{10}$$

for $f \leqq g \leqq h$.

Now we shall consider the congruences θ_h associated with $\chi'(0, -)\varphi'(0h, -)$. These are monomial weakly distributive congruences with the kernel 0 in the sense of Schmidt [6]. Thus, if we prove that $B(0, -)/\bigvee_{h \in H} \theta_h$ is isomorphic to D, then we are done by the following theorem of Schmidt [6]: if θ_h, $h \in H$, are monomial weakly distributive congruences of the generalized Boolean lattice B, then $B/\bigvee_h \theta_h$ is isomorphic to the semilattice of all compact congruences of a lattice.

3. The main construction

We start to define the $D(h, i)$'s. Motivation: Whenever $h \leqq i$, $D(h, i)$ will be a "reduced free product" of all the D_x, $h \leqq x \leqq i$, in the class of distributive lattices with 0, namely we take free 0-product in the class of distributive lattices and factorize it by a congruence (by the smallest possible) so as to insure that in the factor lattice all the relations $d \leqq d\varepsilon(x, y)$, $h \leqq x \leqq y \leqq i$, $d \in D_x$, hold (here, for brevity d etc. stands for the congruence class of d etc.). This free choice of the $D(h, i)$'s is one of the important ideas of the proof, however, we shall not need in the proof that $D(h, i)$ is really free (relative to the given relations), we only need the description given in the following definition.

Definition. $D(h, i)$ will be the finite distributive lattice with the following $\vee$-irreducibles: j is an irreducible of $D(h, i)$ if j is a mapping of a dual segment P to the poset $[h, i]$ to $\bigcup_{x \in P} D_x$ such that for all $x \in P$, j_x is an irreducible of D_x (0 is not irreducible) and whenever $x \leqq y$, $x, y \in P$, then $j_y \leqq j_x \varepsilon(x, y)$.[5])

[5]) They are ordered componentwise, that is $(j_x | x \in P) \leqq (j'_x | x \in Q)$ if $P \supseteqq Q$ and, for all $x \in Q$, $j_x \leqq j'_x$.

Definition. The irreducibles of $D(h, i)$ are the irreducibles of $D(g, i)$, too, if $g \leqq h$. Therefore we get an embedding $\varphi(gh, i)$, if we map the irreducibles $j \in D(h, i)$ to $j \in D(g, i)$ and extend this map such that the $\vee$ is preserved. (This is a lattice embedding as its dual mapping — by Priestley's duality — maps $(j_x | x \in P)$, P a dual segment of $[g, i]$, to $(j_x | x \in P \cap [h, i])$ and therefore is onto).

Definition. $\varphi(h, ij)$ is defined as follows. The irreducibles of $D(h, i)$ are the choice functions $(j_x | x \in P)$ where P is a dual segment of $[h, i]$. Now $(j_x | x \in P)\varphi(h, ij)$ is the join of all $(j'_x | x \in Q)$ such that Q is a dual segment of $[h, j]$, $Q \cap [h, i] = P$, $j'_x \leqq j_x$ for $x \in P$, and $(j'_x | x \in Q)$ is an irreducible in $D(h, j)$. To arbitrary elements of $D(h, i)$ we extend this mapping in such a way that it preserves joins.

Now the commutativities (1), (2), (3) are evident. To show that $\varphi(h, ij)$ is one-to-one we have to prove that its dual mapping is onto. To do that we first describe how the poset $[h, i]$ is obtained from $[h, j]$. According to Figure 1, j is the greatest element of a finite chain in $\{h_\gamma \colon \gamma \leqq \omega(\beta+1)\} \setminus \{h_\gamma \colon \gamma \leqq \omega\beta\}$ for some γ. Omitting this chain we obtain another poset. This remaining poset has a largest element, so we can continue this procedure until the largest element in the remaining poset is i. Now, if we go the other way around, we get $[h, j]$ from $[h, i]$ in such a way, that add finite chains $a_{11}, a_{12}, \ldots, a_{1n_1}; a_{21}, a_{22}, \ldots, a_{2n_2}; \ldots; a_{m1}, a_{m2}, \ldots, a_{mn_m}$ successively to $[h, i]$ as in Figure 2 $(a_{0n_0} = i,\ a_{mn_m} = j)$.

Now we show that the dual map of $\varphi(h, ij)$ is onto. Let $(j_x | x \in P)$ be an irreducible of $[h, i]$, where P is a dual ideal of $[h, i]$. For simplicity, we assume that the adjoined elements are l, m, n, and the chain consisting of the lower covers of these elements is i, h, k.

Now we may choose an irreducible j_l in D_l such that $j_i\varepsilon(i, l) \geqq j_l$. $j_h\varepsilon(h, l) \geqq j_l$, too. Let $x = j_h\varepsilon(h, m)$. Then $x\varepsilon(m, l) \geqq j_l$. x is a join of join-irreducibles: $x = \bigvee_\gamma j_\gamma$ and $j_l \leqq \bigvee_\gamma j_\gamma\varepsilon(m, l)$, thus, for some γ_0, $j_l \leqq j_{\gamma_0}\varepsilon(m, l)$. Define $j_m = j_{\gamma_0}$.

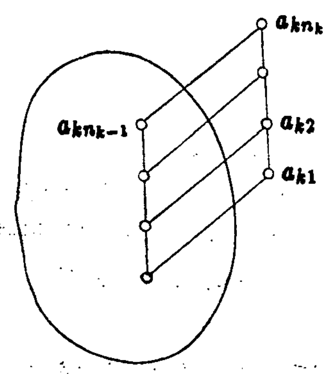

Figure 2

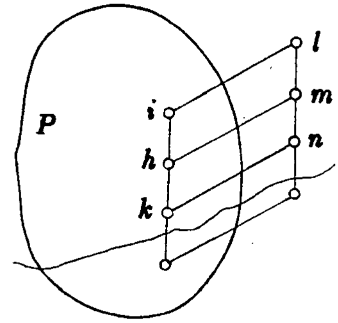

Figure 3

Similarly, we can define j_n, and continuing this procedure we get a vector $(j_x|x\in Q)$ which is mapped to $(j_x|x\in P)$.[6])

To define $B(0, i)$, we agree that the atoms of $B(0, i)$ are $[j_x|x\in P]$ where $(j_x|x\in P)$ is a join-irreducible of $D(0, i)$, the only difference is that in $B(0, i)$ they are, of course, not ordered. The embedding $\chi(0, i)$ is defined as in [5]. Then the commutativity of (4) and (5) is again evident.

Now let $B(0, -)$ be the direct limit of all $B(0, i)$ and $D(h, -)$ be the direct limit of all $D(h, i)$. Then there exist embeddings $\psi(0, i-)$: $B(0, i)\to B(0, -)$ such that the following diagrams are commutative

(11)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h,ij)"] \arrow[dr] & D(h, j) \arrow[d, "\varphi(h,j-)"] \\ & D(h, -) \end{tikzcd}$$

(12)
$$\begin{tikzcd} B(0, i) \arrow[r, "\psi(0,ij)"] \arrow[dr] & B(0, j) \arrow[d, "\psi(0,j-)"] \\ & B(0, -) \end{tikzcd}$$

with $h\leqq i\leqq j$ and $i\leqq j$, respectively. These commutativities make it possible to define embeddings $\varphi(hg, -)$: $D(h, -)\to D(g, -)$ and $\chi(0, -)$ with $g\leqq h$, such that the following diagrams are commutative:

(13)
$$\begin{tikzcd} D(h, i) \arrow[r, "\varphi(h,i-)"] \arrow[d, "\varphi(hg,i)"'] & D(h, -) \arrow[d, "\varphi(hg,-)"] \\ D(g, i) \arrow[r, "\varphi(g,i-)"'] & D(g, -) \end{tikzcd}$$

(14)
$$\begin{tikzcd} D(0, i) \arrow[r, "\varphi(0,i-)"] \arrow[d, "\chi(0,i)"'] & D(0, -) \arrow[d, "\chi(0,-)"] \\ B(0, i) \arrow[r, "\psi(0,i-)"'] & B(0, -) \end{tikzcd}$$

[6]) By means of some additional observation the case that both i and j lie on a chain added in the inductive construction, can be handled similarly.

Also the following diagrams commute:

$$\text{(15)}\qquad \begin{array}{ccc} D(h,i) & \xrightarrow{\varphi(h,i-)} & D(h,-) \\ \uparrow{\scriptstyle \varphi'(gh,i)} & & \uparrow{\scriptstyle \varphi(gh,-)} \\ D(g,i) & \xrightarrow[\varphi(g,i-)]{} & D(g,-) \end{array}$$

$$\text{(16)}\qquad \begin{array}{ccc} D(0,i) & \xrightarrow{\varphi(0,i-)} & D(0,-) \\ \uparrow{\scriptstyle \chi'(0,i)} & & \uparrow{\scriptstyle \chi'(0,-)} \\ B(0,i) & \xrightarrow[\psi(0,i-)]{} & B(0,-) \end{array}$$

Hence it follows that the $\varphi'(gh, -)$ and $\chi'(0, -)$ are weakly distributive monomial congruences and so are their composition. (To show the commutativities of (15) and (16) we have to show the commutativities of (7) and (9), but it is the same as Lemmas 1, 2 in [4].)

Now we can finish the proof as follows. The factor lattice by the congruence $\bigvee_h \theta_h$ is the direct limit of the $D(h, -)$'s relative to the morphisms $\varphi(gh, -)$. Let us denote this limit by F. F has subsemilattices isomorphic to the D_h's. Namely, $D(h, h) \cong D_h$, hence $D(h, h)\varphi(h, h-) \cong D_h$.

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